Analysis I - MATH 5143 - Homework #12 - Fall 2007

Mike Fairchild[1]

Problem 32. Prove that if $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $x_0 \in \text{Int } A$, then for any $v \in \mathbb{R}^n$,

$$\lim_{t \to 0} \frac{f(x_0 + tv) - f(x_0)}{t} = Df_{x_0} \cdot v.$$

<u>Proof</u>: Let f be differentiable at $x_0 \in \text{Int } A$. Then for $h \in \mathbb{R}^n$ we have

$$\lim_{\|h\| \to 0} \frac{\|f(x_0 + h) - f(x_0) - \boldsymbol{D}f(x_0) \cdot h\|}{\|h\|} = 0.$$

Now fix any $v \in \mathbb{R}^n$ and let t be a scalar. Substituting tv for h above,

$$\lim_{\|tv\| \to 0} \frac{\|f(x_0 + tv) - f(x_0) - \boldsymbol{D}f(x_0) \cdot tv\|}{\|tv\|} = 0.$$

But $\|tv\| = |t| \, \|v\|$, and since $\|v\|$ is fixed, we can write $\|tv\| \to 0$ as $t \to 0$. Do this, and rewrite the above as

$$\lim_{t \to 0} \frac{\left\| \frac{t[f(x_0 + tv) - f(x_0)]}{t} - \boldsymbol{D}f(x_0) \cdot tv \right\|}{|t| \, \|v\|} = 0,$$

$$\Rightarrow \lim_{t \to 0} \frac{\cancel{|t|} \left\| \frac{f(x_0 + tv) - f(x_0)}{t} - \boldsymbol{D}f(x_0) \cdot v \right\|}{\cancel{|t|} \, \|v\|} = 0,$$

$$\Rightarrow \lim_{t \to 0} \left\| \frac{f(x_0 + tv) - f(x_0)}{t} - \boldsymbol{D}f(x_0) \cdot v \right\| = 0.$$

But in a normed vector space, the only vector whose magnitude is zero is the zero vector. Hence

$$\lim_{t \to 0} \frac{f(x_0 + tv) - f(x_0)}{t} = \boldsymbol{D}f(x_0) \cdot v.$$

[1] http://www.mikef.org/