Analysis I - MATH 5143 - Homework #8 - Fall 2007
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Problem 15. Do Marsden 4.42 (see quotation block below) except part (a). Work the problem in this order: (d), (c) [quote theorms (do not use the definition) to show that L' exists and is as stated], (b) in the form: for $a, b > 0$, $L(ab) = L(a) + L(b)$, then (e).

> **4.42 (Marsden)** Let $x > 0$, define $L(x) = \int_1^x (1/t)\,dt$. Prove the following, using this definition:
>
> a) L is increasing in x.
>
> b) $L(xy) = L(x) + L(y)$.
>
> c) $L'(x) = 1/x$.
>
> d) $L(1) = 0$.
>
> e) Properties (c) and (d) uniquely determine L. What is L?

<u>Proof</u>:

(d) For $x = 1$ the interval under consideration is $[1, 1] = \{1\}$, a single point. For any partition $\mathcal{P}$, it follows immediately from the definitions that $U(f, \mathcal{P}) = L(f, \mathcal{P}) = 0$ and also that $s = S = 0$. Hence $L(1) = \int_1^1 (1/t)\,dt = 0$.

(c) By Theorem 4.8.8, $L(x) = \int_1^x (1/t)\,dt = F(x) - F(1)$, with F any antiderivative of $f(t) = 1/t$ on $(0, \infty)$. Taking derivatives, $\frac{d}{dx}(L(x)) = \frac{d}{dx}(F(x) - F(1))$. Since $F(1)$ is a constant, $\frac{d}{dx}(F(1)) = 0$, so $\frac{d}{dx}(L(x)) = L'(x) = \frac{d}{dx}(F(x))$. Since F is an antiderivative of f, we have $\frac{d}{dx}(F(x)) = f(x) = 1/x$ according to the paragraph preceding Theorem 4.8.8. Hence $L'(x) = 1/x$, as stated.

(b) Fix $a > 0$. Define $M(x) = L(ax) = \int_1^{ax} (1/t)\,dt$. By Theorems 4.8.8 and 4.7.6, $\frac{d}{dx}[M(x) - L(x)] = \frac{d}{dx}\left[\int_1^{ax} \frac{1}{t}\,dt - \int_1^x \frac{1}{t}\,dt\right] = \frac{1}{ax} \cdot a - \frac{1}{x} = \frac{1}{x} - \frac{1}{x} = 0$. By Corollary 4.7.12, $M(x) - L(x) = K \Rightarrow L(ax) = L(x) + K$ for some constant K. Part (d) implies $L(1) = 0$. Setting $x = 1$, $L(a) = \cancelto{0}{L(1)} + K = K$, so $K = L(a)$. Hence $L(ax) = L(x) + L(a)$, whenever $a, x > 0$. Relabeling, if $a, b > 0$ then $L(ab) = L(a) + L(b)$.

(e) Suppose $F : (0, \infty) \to \mathbb{R}$ and $G : (0, \infty) \to \mathbb{R}$ satisfy (c) and (d). By hypothesis for $x > 0$, $\frac{d}{dx}[G(x) - L(x)] = \frac{1}{x} - \frac{1}{x} = 0$, so by Corollary 4.7.12 $G(x) = F(x) + K$ for a constant K. But $G(1) = F(1) = 0 \Rightarrow 0 = 0 + K \Rightarrow K = 0$. Hence $G(x) = F(x)$, so the function defined by properties (c) and (d) is unique. By this point, it should be clear that $L(x) = \ln(x)$, the natural logarithm function.

Problem 16 (Marsden 4.43). Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and set $F(x) = \int_0^{x^2} f(y)\,dy$. Prove that $F'(x) = 2xf(x^2)$. Give a more general theorem.

<u>Proof</u>: Since f is continuous, by Theorem 4.8.8 an antiderivative F exists such that $F'(x) = f(x)$. Moreover, $F(x) = \int_0^{x^2} f(y)\,dy = F(x^2) - F(0)$ so that $F'(x) = F'(x^2)\frac{d}{dx}x^2 - \frac{d}{dx}F(0) = 2xf(x^2)$ by the Chain Rule (4.7.6). A more general theorem is the following: If $f : \mathbb{R} \to \mathbb{R}$ is continuous, then

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)\,dt = f(v(x))\frac{dv(x)}{dx} - f(u(x))\frac{du(x)}{dx}.$$

This result follows immediately from Theorems 4.8.8 and 4.7.6. This is the single variable version of Leibniz's Rule for differentiating integrals.

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Problem 17. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \notin \{\frac{1}{n} \ : \ n \in \mathbb{N}\} \\ 1 & \text{if } x \in \{\frac{1}{n} \ : \ n \in \mathbb{N}\} \end{cases}.$$

Using the definition of a Riemann integral, show that $\int_0^1 f(x)\,dx$ exists and is 0.

<u>Proof</u>: Let $\mathcal{P} = \{0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1\}$ be any partition of $[0, 1]$. For $i \in \{0, 1, \ldots, n-1\}$, define $\Delta x_i = x_{i+1} - x_i$, $M_i = \sup\{f(x) : x \in [x_i, x_{i+1}]\}$, and $m_i = \inf\{f(x) : x \in [x_i, x_{i+1}]\}$. Hence $U(f, \mathcal{P}) = \sum_{i=0}^{n-1} M_i \Delta x_i$ and $L(f, \mathcal{P}) = \sum_{i=0}^{n-1} m_i \Delta x_i$. Irrationals are dense in $\mathbb{R}$, so any subinterval $[x_i, x_{i+1}]$ contains an irrational. Hence by definition of f and $L(f, \mathcal{P})$, we have $L(f, \mathcal{P}) = 0$ for any $\mathcal{P}$. Thus $s = \sup_{\mathcal{P}}\{L(f, \mathcal{P})\} = 0$. Next we show that $S = \inf_{\mathcal{P}}\{U(f, \mathcal{P})\} = 0$, from which the desired result follows.

Let $\epsilon \in (0, 1)$. Since $\frac{1}{n} \to 0$, there are finitely many $n \in \mathbb{N}$ with $\frac{1}{n} > \frac{\epsilon}{2}$. Let k be the largest natural number such that $\frac{1}{k} > \frac{\epsilon}{2}$. Choose δ so that $0 < \delta < \frac{1}{2}\min\{\frac{1}{k} - \frac{\epsilon}{2}, \frac{1}{k-1} - \frac{1}{k}, \frac{\epsilon}{2+k}\}$. Let $\mathcal{P}' = \{0, \frac{\epsilon}{2}, \frac{\epsilon}{2} + \delta, \frac{1}{k} - \delta, \frac{1}{k} + \delta, \frac{1}{k-1} - \delta, \frac{1}{k-1} + \delta, \ldots, \frac{1}{2} - \delta, \frac{1}{2} + \delta, 1 - \delta, 1\}$. Since $\delta < \frac{1}{2}(\frac{1}{k} - \frac{\epsilon}{2})$ and since $\delta < \frac{1}{2}(\frac{1}{k} - \frac{1}{k-1})$, each element in $\mathcal{P}'$ is strictly greater than the preceding element listed. Moreover, by construction, the only subintervals in $\mathcal{P}'$ containing numbers of the form $\frac{1}{n}$ for some $n \in \mathbb{Z}$ are the subintervals $[0, \frac{\epsilon}{2}]$, $[\frac{\epsilon}{2}, \frac{\epsilon}{2} + \delta]$, subintervals of the form $[\frac{1}{k} - \delta, \frac{1}{k} + \delta]$, and finally the subinterval $[1 - \delta, 1]$. Each of these $2 + k$ intervals has width at most 2δ, so $U(f, \mathcal{P}') \le 1 \cdot 2\delta \cdot (2 + k) < 2\frac{\epsilon}{2(2+k)}(2 + k) = \epsilon$. Hence $0 \le U(f, \mathcal{P}') < \epsilon$, and since ϵ can be chosen arbitrarily close to zero, we see that $S = \inf_{\mathcal{P}}\{U(f, \mathcal{P})\} = 0$. The desired result now follows.

Problem 18 (Marsden 4.44). Let $f : [0, 1] \to \mathbb{R}$ be a bounded and Riemann integrable function, and suppose for every a, b with $0 \le a < b \le 1$ there is a c, $a < c < b$, with $f(c) = 0$. Prove that $\int_0^1 f = 0$. Must f be zero? What if f is continuous?

<u>Proof</u>: Let $\mathcal{P} = \{0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1\}$ be any partition of $[0, 1]$. Define m_i, M_i, and Δx_i as in Problem 17 above. By hypothesis on f, in any subinterval $[x_i, x_{i+1}]$ induced by $\mathcal{P}$, there is at least one point at which $f = 0$. Hence $m_i \le 0$ and $M_i \ge 0$. Thus $L(f, \mathcal{P}) \le 0$ since it is a sum of nonpositive terms, and $U(f, \mathcal{P}) \ge 0$ since it is a sum of nonnegative terms. Since $f : [0, 1] \to \mathbb{R}$ is Riemann integrable, we must have $s = \sup_{\mathcal{P}}\{L(f, \mathcal{P})\} \le 0 = S = \inf_{\mathcal{P}}\{U(f, \mathcal{P})\} \ge 0$. This forces $s = S = 0$. Hence $\int_0^1 f = 0$. Note that f need not be the zero function, since it might be nonzero at a finite number of points. For example, if f is defined as

$$f = \begin{cases} 0 & x \ne \frac{1}{2} \\ 1 & x = \frac{1}{2} \end{cases},$$

then $\int_0^1 f = 0$ but $f \ne 0$. If f is continuous, then we must have $f = 0$. For if $f \ne 0$, then there is at least one point $x_0 \in [0, 1]$ such that $f(x_0) = \alpha \ne 0$. By the given information, we may choose a number $\Delta x > 0$ arbitrarily small such that $f(x + \Delta x) = 0$. Thus $\lim_{x \to x_0^+} f(x) = 0$. Similarly, $\lim_{x \to x_0^-} f(x) = 0$. Hence $\lim_{x \to x_0} f(x) = 0 \ne \alpha = f(x_0)$. But this means f is not continuous at x_0, contradicting the assumption. Hence if f is continuous, we must have $f = 0$.