Abstract Linear Algebra - MATH 5164 - Graded Homework #14 - Spring 2008

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Problem 1 (FIS 7.1.2bd). For each matrix A find a basis for each generalized eigenspace of L_A consisting of a union of disjoint cycles of generalized eigenvectors. Then find a Jordan canonical form J of A.

$$\text{(b)} \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}, \quad \text{(d)} \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 1 & -1 & 3 \end{pmatrix}$$

<u>SOLUTION</u>:

(a) By inspection, the characteristic polynomial of A is $(1-\lambda)(2-\lambda)-6 = \lambda^2 - 3\lambda - 4 = (\lambda-4)(\lambda+1)$ with roots $\lambda_1 = -1$, $\lambda_2 = 4$. We have a full set of distinct eigenvalues, so in fact this matrix can be diagonalized, and such a diagonal representation is in fact a Jordan canonical form. By inspection $N(A-\lambda_1 I) = N\begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} := v_1$ and $\{v_1\}$ is a basis for $E_{\lambda_1} = K_{\lambda_1}$. By inspection $N(A-\lambda_2 I) = N\begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} := v_2$ and $\{v_2\}$ is a basis for $E_{\lambda_2} = K_{\lambda_2}$. Putting $\beta = \{v_1, v_2\}$, a Jordan canonical form of A is $J = [A]_\beta = \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}$.

(b)The characteristic polynomial of A is

$$\begin{vmatrix} 2-\lambda & 1 & 0 & 0 \\ 0 & 2-\lambda & 1 & 0 \\ 0 & 0 & 3-\lambda & 0 \\ 0 & 1 & -1 & 3-\lambda \end{vmatrix} = (3-\lambda)^2(2-\lambda)^2,$$

so $\lambda_1 = 2$ and $\lambda_2 = 3$ are the eigenvalues, both with multiplicity two. A side calculation shows that $\dim(E_{\lambda_1}) = 1$, therefore the basis β_1 for the generalized eigenspace K_{λ_1} consists of a cycle of generalized eigenvectors of length 2 (it could not be 2 cycles of length 1, for that would contradict $\dim(E_{\lambda_1}) = 1$.) We now find a basis for $N((A-\lambda_1 I)^2)$.

$$N((A-\lambda_1 I)^2) = N\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{pmatrix} = N\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right\} := \{v_1, v_2\}.$$

Now, $(A-\lambda_1 I)(v_2) = (1\ 0\ 0\ 0)^t \neq 0$. Hence put $\beta_1 = \{(A-\lambda_1 I)(v_2), v_2\} = \{(1\ 0\ 0\ 0)^t, (0\ 1\ 0\ -1)^t\}$. For $\lambda_2 = 3$, a side calculation shows that $\dim(E_{\lambda_2}) = 2$. Therefore, the basis β_2 for the generalized

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eigenspace K_{λ_2} consists of a single cycle of length 2, and in fact $K_{\lambda_2} = E_{\lambda_2}$.

$$N(A - \lambda_2 I) = N \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix} = N \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} := \{v_3, v_4\}.$$

Therefore, if we put $\beta = \beta_1 \cup \beta_2 = \{v_1, v_2, v_3, v_4\}$, a Jordan canonical form of A is

$$J = [A]_\beta = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

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Problem 2 (FIS 7.1.7abcd). Let U be a linear operator on a finite-dimensional vector space V. Prove the following:

(a) $N(U) \subseteq N(U^2) \subseteq \cdots \subseteq N(U^k) \subseteq N(U^{k+1}) \subseteq \cdots$.

(b) If $\text{rank}(U^m) = \text{rank}(U^{m+1})$ for some positive integer m then $\text{rank}(U^m) = \text{rank}(U^k)$ for any positive integer $k \geq m$.

(c) If $\text{rank}(U^m) = \text{rank}(U^{m+1})$ for some positive integer m then $N(U^m) = N(U^k)$ for any positive integer $k \geq m$.

(d) Let T be a linear operator on V, and let λ be an eigenvalue of T. Prove that if $\text{rank}((T-\lambda I)^m) = \text{rank}((T - \lambda I)^{m+1})$ for some integer m then $K_\lambda = N((T - \lambda I)^m)$.

Proof: I first establish the following useful lemma.

Lemma: Given a linear operator $U : V \to V$ and $m, n \in \mathbb{N}$ with $m \geq n$ then $R(U^m) \subseteq R(U^n)$.

Proof: If $w \in R(U^m)$ then $\exists x \in V$ such that $w = U^m(x) = U^n(U^{m-n}(x))$. By putting $v = U^{m-n}(x)$, we see that $U^n(v) = w$ as well, so $w \in R(U^n)$. Therefore, $R(U^m) \subseteq R(U^n)$. △

(a) If $x \in N(U)$ then $U(x) = 0$ and so $U^2(x) = U(U(x)) = U(0) = 0$ since U is linear. Therefore $N(U) \subseteq N(U^2)$. The result follows by a trivial induction.

(b) By hypothesis, $\dim(R(U^m)) = \text{rank}(U^m) = \text{rank}(U^{m+1}) = \dim(R(U^{m+1}))$. By the lemma, $R(U^{m+1}) \subseteq R(U^m)$, so by Theorem 1.11, $R(U^m) = R(U^{m+1})$. By induction then, $R(U^k) = R(U^m)$ for any $k \geq m$. Therefore, $\text{rank}(U^k) = \dim(R(U^k)) = \dim(R(U^m)) = \text{rank}(U^m)$.

(c) By (b), $\text{rank}(U^m) = \text{rank}(U^k)$ for $k \geq m$. By the Dimension Theorem (2.3), we must have $\dim(N(U^m)) = \dim(N(U^k))$. By part (a), $N(U^m) \subseteq \cdots \subseteq N(U^k)$, and the result now follows by Theorem 1.11.

(d) By (a), $N((T-\lambda I)^k)$ is an increasing family of sets. But by (c), $N((T-\lambda I)^m) = N((T-\lambda I)^k)$ for all $k \geq m$. So $\cup_{q \in \mathbb{N}} N((T - \lambda I)^q) = N((T - \lambda I)^m)$. Now, $K_\lambda = \{x \in V : (T - \lambda I)^p(x) = 0\} = \cup_{q \in \mathbb{N}} N((T - \lambda I)^q) = N((T - \lambda I)^m)$. ■