Abstract Linear Algebra - MATH 5164 - Graded Homework #4 - Spring 2008

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Problem 1 (FIS 2.3.9). Find linear transformations $U, T : F^2 \to F^2$ such that $UT = T_0$ (the zero transformation) but $TU \neq T_0$. Use your answer to find matrices A and B such that $AB = 0$ but $BA \neq 0$.

<u>Proof</u>: A little experimentation with pencil and paper produced the following simple example. Let $T(a, b) = (0, a)$ and $U(a, b) = (a, 0)$. T is a flip and then projection and U is just a projection. Then, $(UT)(a, b) = U(T(a, b)) = U(0, a) = (0, 0)$, so $UT = T_0$, whereas $(TU)(a, b) = T(U(a, b)) = T(a, 0) = (0, a) \neq 0$ whenever $a \neq 0$. So $TU \neq T_0$. If we work in the standard bases $\beta = \{(1, 0), (0, 1)\}$ of F^2 and denote $[T]_\beta = A$ and $[U]_\beta = B$, then

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \ BA = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \ \blacksquare$$

Problem 2 (FIS 2.3.10). Let A be an $n \times n$ matrix. Prove that A is a diagonal matrix if and only if $A_{ij} = \delta_{ij} A_{ij}$ for all i and j.

<u>Proof</u>:

($\Rightarrow$) Suppose $A_{ij} = \delta_{ij} A_{ij}$. Then $A_{ij} = 0$ whenever $i \neq j$ by definition of δ_{ij}. But these entries are precisely the off-diagonal entries, so A can be nonzero only along its main diagonal. Hence A is diagonal.

($\Leftarrow$) Suppose A is an $n \times n$ diagonal matrix. We consider each element A_{ij} for all i and j. There are two cases. Either $i \neq j$ or $i = j$. If $i \neq j$ we must have $A_{ij} = 0$ since A is diagonal by hypothesis. Thus $A_{ij} = 0 = \delta_{ij} A_{ij}$ since $\delta_{ij} = 0$ whenever $i \neq j$. If $i = j$ then $A_{ij} = \delta_{ij} A_{ij}$, since $\delta_{ij} = 1$ whenever $i = j$. So in either case, $A_{ij} = \delta_{ij} A_{ij}$. $\blacksquare$

Problem 3 (FIS 2.3.12). Let V, W, and Z be vector spaces, and let $T : V \to W$ and $U : W \to Z$ be linear.

(a) Prove that if UT is one-to-one, then T is one-to-one. Must U also be one-to-one?
(b) Prove that if UT is onto, then U is onto. Must T also be onto?
(c) Prove that if U and T are one-to-one and onto, then UT is also.

<u>Proof</u>:

(a) Assume T is not one-to-one. Then there exist $x, y \in V$ such that $T(x) = T(y)$. But then $(UT)(x) = U(T(x)) = U(T(y)) = (UT)(y)$, contradicting the fact that UT is one-to-one. Hence T must be one-to-one. U need not be one-to-one. Here is a counterexample. Let $V = \mathbb{R}, W = \mathbb{R}^2, Z = \mathbb{R}$, and define $T : V \to W$ and $U : W \to Z$ by $T(x) = (x, 0)$ and $U(x, y) = x + y$. Then it's clear that UT and T are one-to-one, but U is certainly not one-to-one (*e.g.* $U(x, y) = U(y, x)$).

(b) Assume UT is onto. Then for any $z \in Z$ there exists a $v \in V$ such that $(UT)(v) = U(T(v)) = z$. Clearly $w = T(v) \in W$, so for this w we have $U(w) = z$. Hence U is onto. T

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need not be onto. Here is a counterexample. Let $V = \mathbb{R}, W = \mathbb{R}^2, Z = \mathbb{R}$. Define $T : V \to W$ and $U : W \to Z$ by $T(x) = (x, 0)$ and $U(x, y) = x$. Clearly UT and U are onto, but T is not onto (there is no $x \in V$ that maps to $(x, 1)$, say).

(c) Assume both U and T are one-to-one. Since T is one-to-one, then for $x, y \in V$ with $x \neq y$ we have $T(x) \neq T(y)$. And since U is one-to-one, $(UT)(x) = U(T(x)) \neq U(T(y)) = (UT)(y)$, which shows that UT is one-to-one. Now assume both U and T are onto. Choose any $z \in Z$. Since U is onto, there is a $w \in W$ such that $U(w) = z$. Since T is onto, there is a $v \in V$ such that $T(v) = w$. So for this v, $(UT)(v) = U(T(v)) = U(w) = z$, showing that UT is onto. ■

Problem 4 (FIS 2.3.13). Let A and B be $n \times n$ matrices. Recall that the trace of A is defined by

$$\operatorname{tr}(A) = \sum_{i=1}^{n} A_{ii}.$$

Prove that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ and $\operatorname{tr}(A) = \operatorname{tr}(A^t)$.

<u>Proof</u>: Recall that $(AB)_{ij} = \sum_k A_{ik}B_{kj}$. So,

$$\operatorname{tr}(AB) = \sum_i (AB)_{ii} = \sum_i \sum_k A_{ik}B_{ki} = \sum_i \sum_k B_{ki}A_{ik} = \sum_i \sum_k B_{ik}A_{ki} = \sum_i (BA)_{ii} = \operatorname{tr}(BA)$$

where the fourth equality holds because summing over all choices of i, k permits us to switch indices (since all i, k pairs will eventually be included, regardless of order). That $\operatorname{tr}(A) = \operatorname{tr}(A^t)$ is obvious, for diagonal elements are invariant under transposition ($A_{ij} \leftrightarrow A_{ji}$ doesn't change anything when $i = j$). ■

Problem 5 (FIS 2.4.5). Let A be invertible. Prove that A^t is invertible and $(A^t)^{-1} = (A^{-1})^t$.

<u>Proof</u>: Assume A is invertible. Then there is a matrix $B = A^{-1}$ such that $AB = I = BA$. So,

$$\delta_{ij} = I_{ij} = (AB)_{ij} = \sum_k A_{ik}B_{kj} = \sum_k A^t_{ki}B^t_{jk} = \sum_k B^t_{jk}A^t_{ki} = \sum_k (A^{-1})^t_{jk}A^t_{ki},$$

which shows that $(A^{-1})^t A^t = I$, proving the claim. ■

Problem 6 (FIS 2.4.6). Prove that if A is invertible and $AB = 0$, then $B = 0$.

<u>Proof</u>: If A is invertible, then A^{-1} exists such that $A^{-1}A = I = AA^{-1}$. So left multiply each side of $AB = 0$ by A^{-1} to get $AA^{-1}B = A^{-1}0 \Rightarrow IB = 0 \Rightarrow B = 0$. ■