Complex Analysis I - MATH 8141 - Fall 2007 - Stereographic Projection

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We show there is a one-to-one correspondence between the complex plane $\mathbb{C}$ and the sphere $\mathcal{S} = S^2\backslash\{(0,0,1)\} = \{(x,y,z) : x^2+y^2+z^2 = 1\}\backslash\{(0,0,1)\}$ (*i.e.* the surface of the unit sphere in $\mathbb{R}^3$ without the north pole $N = (0,0,1)$). We consider the complex plane to be embedded in $\mathbb{R}^3$, that is, $\mathbb{C} = \{(x,y,0) \in \mathbb{R}^3 : x,y \in \mathbb{R}\}$.

Let $z = (x,y) \in \mathbb{C}$. If $L : [0,1] \to \mathbb{R}^3$ is the parametric path in $\mathbb{R}^3$ from z_0 to $N = (0,0,1)$, then

$$L(t) = ((1-t)x, (1-t)y, t) \text{ for } 0 \le t \le 1. \tag{1}$$

By definition of $\mathcal{S}$, the image of L intersects $\mathcal{S}$ at a point $s = (\xi, \eta, \zeta)$ when t is chosen so that $(1-t)^2x^2 + (1-t)^2y^2 + t^2 = 1$. We can expand and rearrange this equation to $(x^2+y^2+1)t^2 + (-2x^2-2y^2)t + (x^2+y^2-1) = 0$. This is a quadratic equation in t whose roots $t_\pm$ are given by

$$\begin{aligned} t_\pm &= \frac{(2x^2+2y^2) \pm \sqrt{(-2x^2-2y^2)^2 - 4(x^2+y^2+1)(x^2+y^2-1)}}{2(x^2+y^2+1)} \\ &= \frac{(2x^2+2y^2) \pm \sqrt{4}}{2(x^2+y^2+1)} \\ &= \frac{x^2+y^2 \pm 1}{x^2+y^2+1} = \frac{|z|^2 \pm 1}{|z|^2+1} \end{aligned}$$

Note that $t_+ = 1$, in which case $s = (0,0,1) = N \notin \mathcal{S}$, so we are only interested in the other solution t_-. Thus define $t_{\mathcal{S}} = t_- = \frac{x^2+y^2-1}{x^2+y^2+1}$, since this is the value of t that, when substituted into Eq. (1), gives a point on $\mathcal{S}$. If we do the substitution, we find that

$$s = (\xi, \eta, \zeta) = \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1}\right) = \frac{1}{x^2+y^2+1}(2x, 2y, x^2+y^2-1). \tag{2}$$

Now suppose we are given a point $s = (\xi, \eta, \zeta) \in \mathcal{S}$ and want to find the corresponding point $z = (x,y) \in \mathbb{C}$. Let $L_\star : \mathbb{R} \to \mathbb{R}^3$ denote the infinite (directed) line from N to $(x,y,0)$. Parametrically,

$$L_\star(t) = (t\xi, t\eta, t(\zeta-1)+1) \text{ for } -\infty < t < +\infty. \tag{3}$$

Note that $L_\star(0) = (0,0,1) = N$ and $L_\star(1) = s = (\xi, \eta, \zeta)$. That is, the image of $L_\star$ indeed joins the north pole N and the point s. We want to determine where the image of $L_\star$ intersects the complex plane, which happens when $z = 0$. To achieve $z = 0$, we need $t(\zeta-1)+1 = 0$ which happens when $t = t_{\mathbb{C}} = \frac{-1}{\zeta-1} = \frac{1}{1-\zeta}$. Substituting $t_{\mathbb{C}}$ into Eq. (3), we get

$$z = (x,y,0) = \left(\frac{\xi}{1-\zeta}, \frac{\eta}{1-\zeta}, 0\right) = \frac{1}{1-\zeta}(\xi, \eta, 0).$$

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