The Snowplow Problem[1]

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Question:

One day it started snowing at a heavy and steady rate. A snowplow started out at noon, going 2 miles the first hour and 1 mile the second hour. What time did it start snowing?

Solution:

Of course we must make an assumption to move forward in this problem. The critical assumption is that the speed of the snowplow is inversely proportional to the depth of the snow that it's plowing. With that as the only very general assumption, we can solve this problem. Let us proceed as follows.

Define h to be the height of the snow. But observe that the height of the snow is directly proportional to the time, since it's falling at a constant rate, which I'll call r. Thus, $h = rt$ and we can write:

$$\frac{dx}{dt} \propto \frac{1}{h} \propto \frac{1}{rt} = \frac{k}{rt} = \frac{A}{t}$$

where I have absorbed k/r into the constant A. This is a straightforward separable first order differential equation which, after separating the variables, looks like this:

$$\frac{dx}{A} = \frac{dt}{t}.$$

Integrating both sides goes as follows:

$$\int \frac{dx}{A} = \int \frac{dt}{t}$$

$$\frac{x}{A} + C_1 = \ln t + C_2$$

$$x(t) = A \ln t + K$$

[1]This problem was brought to my attention from Dr. Tom R. Lucas of the Department of Mathematics at the University of North Carolina at Charlotte.

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where A has been multiplied to the right hand side and all the constants have been absorbed into K. Now, this problem also has some initial conditions that we must satisfy.

Since the domain of $\ln t$ is $t > 0$, let us define $t = 0$ to be the time when the snow began and $t = \tau$ the time when the snowplow started (noon). Also note that times are measured in hours and distances in miles. Here are the initial conditions we have from the problem:

$$x(\tau) = 0 \tag{1}$$
$$x(\tau + 1) = 2 \tag{2}$$
$$x(\tau + 2) = 3 \tag{3}$$

The first initial condition (1) allows us to solve for K as follows:

$$x(\tau) = 0 = A \ln \tau + K \quad \Rightarrow \quad K = -A \ln \tau$$

giving us, after substitution for K, a complete formulation for $x(t)$ now as follows:

$$x(t) = A \ln t - A \ln \tau = A(\ln t - \ln \tau) = A \ln \left(\frac{t}{\tau} \right).$$

Given $x(t)$ as above and initial conditions (2) and (3), we are now able to eliminate A from the equations and get a solution for τ as follows:

$$x(\tau + 1) = 2 = A \ln \left(\frac{\tau + 1}{\tau} \right) \quad \Rightarrow \quad A = \frac{2}{\ln \left(\frac{\tau+1}{\tau} \right)}$$

and substitution of A into (3) gives us:

$$x(\tau + 2) = 3 = \frac{2}{\ln \left(\frac{\tau+1}{\tau} \right)} \ln \left(\frac{\tau + 2}{\tau} \right) \quad \Rightarrow \quad \frac{3}{2} = \frac{\ln \left(\frac{\tau+2}{\tau} \right)}{\ln \left(\frac{\tau+1}{\tau} \right)}$$

Cross multiplying and exponentiating results in:

$$3 \ln \frac{\tau + 1}{\tau} = 2 \ln \frac{\tau + 2}{\tau} \quad \Rightarrow \quad \left(\frac{\tau + 1}{\tau} \right)^3 = \left(\frac{\tau + 2}{\tau} \right)^2.$$

Now, after doing a few algebra steps, the cubic turns into a quadratic, and after invoking the quadratic forumla we arrive at the following solution:

$$\tau = \frac{-1 + \sqrt{5}}{2}$$

where we disregarded the $-$ in the normal $\pm$ form because we must meet the restriction $\tau > 0$. Therefore, $\tau \approx 0.618$ hours, which is ≈ 37 minutes and the snow must have started at $\approx$ 11:23am. Two things strike me as remarkable about this problem. First, that we can solve a seemingly impossible problem by making a single careful assumption and using the power of differential equations. Secondly, that the analytic form of the solution resembled very closely the golden ratio (ϕ).